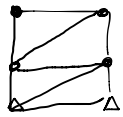
Structural Vibrations



point masses connected by springs

Being dumb about it: For each node $\vec{F} = m\vec{a}$

$$\vec{F}_7 = m_7 \vec{a}_7$$

$$\vec{F} = \sum \text{Forces from springs}$$

Look at one spring: spring from node 5 to node 2

Force on mass ② from mass ⑤: $\vec{F} = -\Delta\ell \frac{\vec{r}}{|\vec{r}|} = -K(\Delta\ell)\,\hat{\lambda}$, $\hat{\lambda} = \frac{\vec{r}}{|\vec{r}|}$

$$\vec{F} = -K\left[|\vec{r}| - |\vec{r}_0|\right]\frac{\vec{r}}{|\vec{r}|}$$

$$= -K\left[|\vec{r}_0 + \Delta\vec{r}| - |\vec{r}_0|\right)\frac{\vec{r}_0 + \Delta\vec{r}}{|\vec{r}_0 + \Delta\vec{r}|}$$

$$= -K\left[\sqrt{(\vec{r}_0 + \Delta\vec{r})\cdot(\vec{r}_0 + \Delta\vec{r})} - \sqrt{\vec{r}_0 \cdot \vec{r}_0}\right]\frac{\vec{r}_0 + \Delta\vec{r}}{\sqrt{(\vec{r}_0 + \Delta\vec{r})\cdot(\vec{r}_0 + \Delta\vec{r})}}$$

$$= -K\left[\sqrt{\vec{r}_0\cdot\vec{r}_0 + 2\Delta\vec{r}\cdot\vec{r}_0 + \Delta\vec{r}\cdot\Delta\vec{r}} - \sqrt{\vec{r}_0\cdot\vec{r}_0}\right]\frac{\vec{r}_0 + \Delta\vec{r}}{\sqrt{\vec{r}_0\cdot\vec{r}_0 + 2\Delta\vec{r}\cdot\vec{r}_0 + \Delta\vec{r}\cdot\Delta\vec{r}}}$$

$$= -K\left[\sqrt{1 + 2\frac{\Delta\vec{r}\cdot\vec{r}_0}{|\vec{r}_0|^2} + \frac{\Delta\vec{r}\cdot\Delta\vec{r}}{|\vec{r}_0|^2}} - 1\right]\cdot\frac{\vec{r} + \Delta\vec{r}}{\sqrt{1 + \frac{2\Delta\vec{r}\cdot\vec{r}_0}{|\vec{r}_0|^2} + |\Delta\vec{r}|^2}}$$

if $\frac{|\Delta\vec{r}|}{|\vec{r}_0|} << 1$

$$\sqrt{1+\varepsilon} \approx 1 + \varepsilon/2$$

$$\frac{1}{1+\varepsilon} \approx 1 + \varepsilon$$

only keep first order stuff

$$= -K\left[1 + \frac{\Delta\vec{r}\cdot\vec{r}_0}{|\vec{r}_0|^2} + \ldots - 1\right]\cdot(\vec{r} + \Delta\vec{r})\left(1 - \frac{\Delta\vec{r}\cdot\vec{r}_0}{|\vec{r}_0|^2} + \ldots\right)$$

and then only keep highest order terms

$$\vec{F} = -K\frac{\Delta\vec{r}\cdot\vec{r}_0}{|\vec{r}_0|^2}\vec{r}_0 = -K\,\hat{\lambda}_0(\hat{\lambda}_0\cdot\Delta\vec{r})$$

$$\vec{F} = -K\left[\hat{\lambda}_0\right]\left[\hat{\lambda}_0\right]^T\left[\Delta\vec{r}\right] \quad \rightarrow \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix}\cdot\begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}^T\begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

$$= -K\left[\left[\hat{\lambda}_0\right]\left[\hat{\lambda}_0\right]^T\right]\left[\Delta\vec{r}\right]$$

$$= \left[\hat{\lambda}\right]\left[\hat{\lambda}\right]^T = \begin{bmatrix} \lambda_x \\ \lambda_y \end{bmatrix}\cdot\begin{bmatrix} \lambda_x & \lambda_y \end{bmatrix}$$

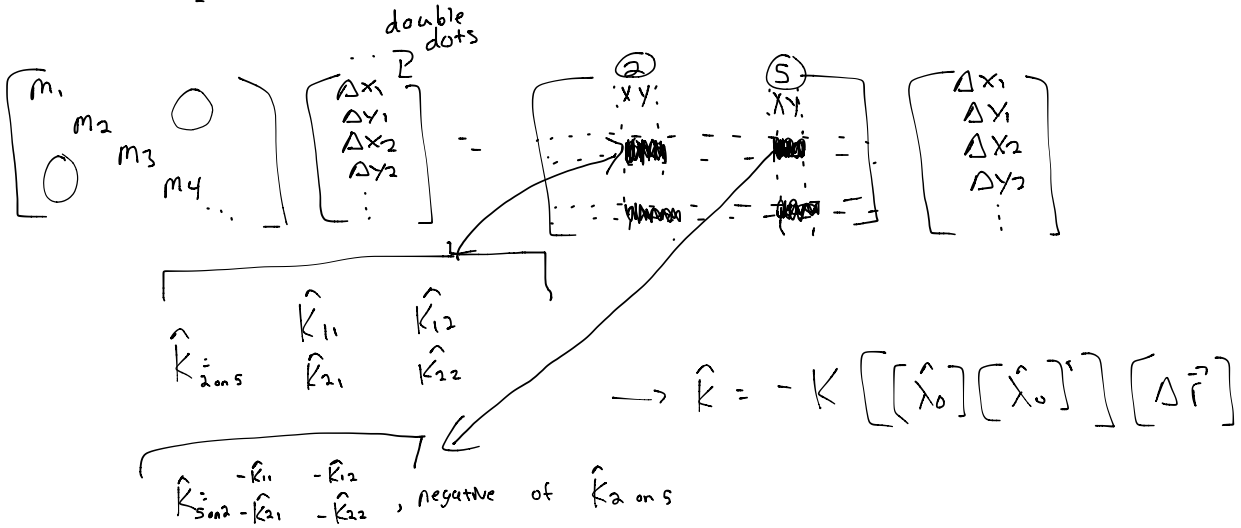
$$= \begin{bmatrix} \lambda_{xx} & \lambda_{xy} \\ \lambda_{xy} & \lambda_{yy} \end{bmatrix} \rightarrow \lambda_x = \cos\theta$$

$$\lambda_y = \sin\theta$$

$$m_2\begin{bmatrix} \ddot{x}_2 \\ \ddot{y}_2 \end{bmatrix} = -K\begin{bmatrix} \lambda_{0x}^2 & \lambda_{0x}\lambda_{0y} \\ \lambda_{0x}\lambda_{0y} & \lambda_{0y}^2 \end{bmatrix}\begin{bmatrix} \Delta x_2 - \Delta x_5 \\ \Delta y_2 - \Delta y_5 \end{bmatrix} \rightarrow \left[\Delta\vec{r}\right] = \begin{bmatrix} \Delta x_2 - \Delta x_5 \\ \Delta y_2 - \Delta y_5 \end{bmatrix}$$

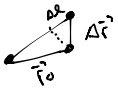
Put together in system equations:

$$M\Delta\ddot{\vec{x}} = -K(\Delta\vec{x})$$



"Assembly" of $\hat{K}$, add K from each spring

Last class review:



$$\Delta l \approx \frac{\Delta\vec{r}\cdot\vec{r}_0}{|\vec{r}_0|} \qquad \vec{F} = -k\frac{\Delta\vec{r}\cdot\vec{r}_0}{|\vec{r}_0|}\frac{\vec{r}_0}{|\vec{r}_0|}$$

$$= -K\,\hat{\lambda}_0\hat{\lambda}_0\cdot\Delta\vec{r}$$